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A generalized statistical convergence via ideals

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ABSTRACT

In this paper we make a new approach to the notions of $[V,\lambda]$-summability and λ-statistical convergence by using ideals and introduce new notions, namely, $I-[V,\lambda]$-summability and $I-\lambda$-statistical convergence. We mainly examine the relation between these two new methods as also the relation between $I-\lambda$-statistical convergence and I-statistical convergence introduced by the authors recently. We carry out the whole investigation in normed linear spaces.

1. Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [1] (see, also [2]) as follows: if N denotes the set of natural numbers and $K \subset N$ then $K(m,n)$ denotes the cardinality of $K \cap [m,n]$. The upper and the lower natural density of the subset K are defined by

$\overline{d}(K) = limsup_{n\to\infty} \frac{K(1,n)}{n}$ and $\underline{d}(K) = liminf_{n\to\infty} \frac{K(1,n)}{n}$.

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence (x_n) of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{n \in N : |x_n - L| \geq \varepsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [3] and Salat [4].

The notion of ideal convergence was introduced first by Kostyrko et al. [5] as a generalization of statistical convergence [1,2] which was further studied in topological spaces [6]. More applications of ideals can be found in [7,6]. In another direction the idea of λ-statistical convergence was introduced and studied by Mursaleen [8] as an extension of the $[V,\lambda]$ summability of Leindler [9]. λ-statistical convergence is a special case of more general A-statistical convergence studied by Kolk in [10].

In this note we intend to unify these two approaches and use ideals to introduce the concept of $I-\lambda$-statistical convergence in line of our recent work [11], and investigate some of its consequences.

Throughout $(X, \|.\|)$ will stand for a real normed linear space and by a sequence $x = (x_n)$ we shall mean a sequence of elements of X. N will stand for the set of natural numbers.

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2. Main results

A family $I \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if (**i**) $\emptyset \in I$; (**ii**) $A, B \in I$ imply $A \cup B \in I$; (**iii**) $A \in I, B \subset A$ imply $B \in I$, while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. If I is an ideal in Y then the collection $F(I) = \{M \subset Y; M^c \in I\}$ forms a filter in Y which is called the filter associated with I.

Let $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to I [5].

Definition 2.1. A sequence $x = (x_k)$ is said to be I-statistically convergent to $L \in X$, if for every $\epsilon > 0$, and every $\delta > 0$,

$$\left\{n \in N : \frac{1}{n}|\{k \leq n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in I.$$

For $I = I_{fin}$, I-statistical convergence coincides with statistical convergence.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$

The collection of such a sequence λ will be denoted by Δ.

The generalized de la Valée–Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be $I - [V, \lambda]$- summable to $L \in X$, if

$$I - lim_n t_n(x) \to L$$

i.e. for any $\delta > 0$,

$$\{n \in N : |t_n(x) - L| \geq \delta\} \in I.$$

If $I = I_{fin}$, $I - [V, \lambda]$-summability becomes $[V, \lambda]$ summability [9].

We now introduce our main definition.

Definition 2.2. A sequence $x = (x_k)$ is said to be $I - \lambda$-statistically convergent or $I - S_\lambda$ convergent to L, if for every $\epsilon > 0$ and $\delta > 0$,

$$\left\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\} \| \geq \delta\right\} \in I.$$

In this case we write $I - S_\lambda - \lim x = L$ or $x_k \to L\ (I - S_\lambda)$. We also write $I - lim\|x_k\| = \|L\|$. For $I = I_{fin}$, $I - S_\lambda$-convergence again coincides with λ-statistical convergence.

We shall denote by $S(I)$, $S_\lambda(I)$ and $[V, \lambda](I)$ the collections of all I-statistically convergent, $I - S_\lambda$-convergent and $I - [V, \lambda]$-convergent sequences respectively.

Theorem 2.1. *Let $\lambda = (\lambda_n) \in \Delta$. Then*

(i) *$x_n \to L[V, \lambda](I) \Rightarrow x_k \to L(S_\lambda(I))$ and the inclusion $[V, \lambda](I) \subset S_\lambda(I)$ is proper for every ideal I.*
(ii) *If $x \in m(X)$, the space of all bounded sequences of X and $x_k \to L\ (S_\lambda(I))$ then $x_k \to L\ [V, \lambda](I)$.*
(iii) *$S_\lambda(I) \cap m(X) = [V, \lambda](I) \cap m(X)$.*

Proof. (i) Let $\epsilon > 0$ and $x_k \to L[V, \lambda](I)$. We have

$$\sum_{k\epsilon I_n} \|x_k - L\| \geq \sum_{k\in I_n \& \|x_k - L\| > \varepsilon} \|x_k - L\| \geq \epsilon . |\{k \in I_n : \|x_k - L\| \geq \epsilon\}|.$$

So for a given $\delta > 0$,

$$\frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta \Rightarrow \frac{1}{\lambda_n}\sum_{k\epsilon I_n} \|x_k - L\| \geq \epsilon\delta$$

i.e. $\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta\} \subset \{n \in N : \frac{1}{\lambda_n}\{\sum_{k\epsilon I_n} \|x_k - L\| \geq \epsilon\} \geq \epsilon\delta\}$.

Since $x_k \to L[V, \lambda](I)$, so the set on the right-hand side belongs to I and so it follows that $x_k \to L(S_\lambda(I))$.

To show that $S_\lambda(I) \subsetneq [V,\lambda](I)$, take a fixed $A \in I$. Define $x = (x_k)$ by

$$x_k = \begin{cases} ku & \text{for } n - [\sqrt{\lambda_n}] + 1 \le k \le n,\ n \notin A \\ ku & \text{for } n - \lambda_n + 1 \le k \le n,\ n \in A \\ \theta & \text{otherwise.} \end{cases}$$

where $u \in X$ is a fixed element with $\|u\| = 1$, and θ is the null element of X. Then $x \notin m(X)$ and for every $\epsilon > 0$ $(0 < \epsilon < 1)$ since

$$\frac{1}{\lambda_n}|\{k \in I_n : \|x_k - 0\| \ge \epsilon\}| = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0$$

as $n \to \infty$ and $n \notin A$, so for every $\delta > 0$,

$$\left\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - 0\| \ge \epsilon\}| \ge \delta\right\} \subset A \cup \{1, 2, \dots, m\}$$

for some $m \in N$. Since I is admissible, it follows that $x_k \to \theta(S_\lambda(I))$. Obviously

$$\frac{1}{\lambda_n}\sum_{k \in I_n} \|x_k - \theta\| \to \infty \quad (n \to \infty)$$

i.e. $x_k \nrightarrow \theta[V,\lambda](I)$. Note that if $A \in I$ is infinite then $x_k \nrightarrow \theta(S_\lambda)$. This example also shows that $I - \lambda$-statistical convergence is more general than λ-statistical convergence.

(ii) Suppose that $x_k \to L(S_\lambda(I))$ and $x \in l_\infty$. Let $\|x_k - L\| \le M\ \forall k$. Let $\epsilon > 0$ be given. Now

$$\begin{aligned} \frac{1}{\lambda_n}\sum_{k \in I_n} \|x_k - L\| &= \frac{1}{\lambda_n}\sum_{k \in I_n \& \|x_k - L\| \ge \varepsilon} \|x_k - L\| + \frac{1}{\lambda_n}\sum_{k \in I_n \& \|x_k - L\| < \varepsilon} \|x_k - L\| \\ &\le \frac{M}{\lambda_n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}| + \epsilon. \end{aligned}$$

Note that $\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}| \ge \frac{\epsilon}{M}\} = A(\epsilon)$ (say) $\in I$. If $n \in (A(\epsilon))^c$ then

$$\frac{1}{\lambda_n}\sum_{k \in I_n} \|x_k - L\| < 2\epsilon.$$

Hence

$$\left\{n \in N : \frac{1}{\lambda_n}\sum_{k \in I_n} \|x_k - L\| \ge 2\epsilon\right\} \subset A(\epsilon)$$

and so belongs to I. This shows that $x_k \to L[V,\lambda](I)$.

(iii) This readily follows from (i) and (ii). $\square$

Theorem 2.2. (i) $S(I) \subset S_\lambda(I)$ *if* $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$.

(ii) *If* $\liminf_{n\to\infty} \frac{\lambda_n}{n} = 0$, *I-strongly (by which we mean that* $\exists$ *a subsequence* $(n(j))_{j=1}^{\infty}$, *for which* $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$ $\forall j$ *and* $\{n(j) : j \in N\} \notin I$*) then* $S(I) \subsetneq S_\lambda(I)$.

Proof. (i) For given $\epsilon > 0$,

$$\begin{aligned} \frac{1}{n}|\{k \le n : \|x_k - L\| \ge \epsilon\}| &\ge \frac{1}{n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}| \\ &\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}|. \end{aligned}$$

İf $\liminf_{n\to\infty} \frac{\lambda_n}{n} = a$ then from definition $\{n \in N : \frac{\lambda_n}{n} < \frac{a}{2}\}$ is finite. For $\delta > 0$,

$$\begin{aligned} \left\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}| \ge \delta\right\} &\subset \left\{n \in N : \frac{1}{n}|\{k \in I_n : \|x_k - L\| \ge \epsilon\}| \ge \frac{a}{2}\delta\right\} \\ &\quad \cup \left\{n \in N : \frac{\lambda_n}{n} < \frac{a}{2}\right\}. \end{aligned}$$

Since I is admissible, the set on the right-hand side belongs to I and this completed the proof of (i).

(ii) Define a sequence $x = (x_j)$ by

$$x_i = \begin{cases} u & \text{if } i \in I_{n(j)},\ j = 1, 2, \ldots \\ \theta & \text{otherwise.} \end{cases}$$

where as before $u \in X$, $\|u\| = 1$ and θ is the zero element of X. Then x is statistically convergent and so $x \in S(I)$ (Since I is admissible). But $x \notin [V, \lambda](I)$ and so by Theorem 2.1(ii) $x \notin S_\lambda(I)$. $\square$

Theorem 2.3. *If $\lambda \in \triangle$ be such that $lim_n \frac{\lambda_n}{n} = 1$, then $S_\lambda(I) \subset S(I)$.*

Proof. Let $\delta > 0$ be given. Since $lim_n \frac{\lambda_n}{n} = 1$, we can choose $m \in N$ such that $|\frac{\lambda_n}{n} - 1| < \frac{\delta}{2}$, for all $n \geq m$. Now observe that, for $\varepsilon > 0$

$$\begin{aligned} \frac{1}{n}|\{k \leq n : \|x_k - L\| \geq \epsilon\}| &= \frac{1}{n}|\{k \leq n - \lambda_n : \|x_k - L\| \geq \epsilon\}| + \frac{1}{n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \\ &\leq \frac{n - \lambda_n}{n} + \frac{1}{n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \\ &\leq 1 - \left(1 - \frac{\delta}{2}\right) + \frac{1}{n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \\ &= \frac{\delta}{2} + \frac{1}{n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}|, \end{aligned}$$

for all $n \geq m$. Hence

$$\left\{n \in N : \frac{1}{n}|\{k \leq n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \subset \left\{n \in N : \frac{1}{n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \frac{\delta}{2}\right\} \cup \{1, 2, 3, \ldots, m\}.$$

If $I - S_\lambda - limx = L$ then the set on the right-hand side belongs to I and so the set on the left-hand side also belongs to I. This shows that $x = (x_k)$ is I-statistically convergent to L. $\square$

Remark 1. We do not know whether the condition in Theorem 2.3 is necessary and leave it as an open problem.

Remark 2. Taking the sequence (λ_n) where $\lambda_n = 1$ for $n = 1$ to 10 and $\lambda_n = n - 10$ for all $n \geq 10$, if we construct the sequence as in Theorem 2.1.(i) and take $I = I_d$ (the ideal of density zero sets of N) then for $A = \{1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$, the sequence $x = (x_k)$ is an example of a sequence which is I-statistically convergent (by Theorem 2.3) but is not statistically convergent.

Theorem 2.4. *$S_\lambda(I) \cap m(X)$ is a closed subset of $m(X)$ if X is a Banach space.*

Proof. Suppose that $(x^n) \subset S_\lambda(I) \cap m(X)$ is a convergent sequence and it converges to $x \in m(X)$. We need to show that $x \in S_\lambda(I) \cap m(X)$. Assume that $x^n \to L_n(S_\lambda(I))\ \forall n \in N$. Take a sequence $\{\epsilon_n\}_{n \in N}$ of strictly decreasing positive numbers converging to zero. We can find an $n \in N$ such that $\|x - x^j\|_\infty < \frac{\epsilon_n}{4}$ for all $j \geq n$. Choose $0 < \delta < \frac{1}{5}$.

Now

$$A = \left\{m \in N : \frac{1}{\lambda_m}\left|\left\{k \in I_m : \|x_k^n - L_n\| \geq \frac{\epsilon_n}{4}\right\}\right| < \delta\right\} \in F(I)$$

and

$$B = \left\{m \in N : \frac{1}{\lambda_m}\left|\left\{k \in I_m : \|x_k^{n+1} - L_{n+1}\| \geq \frac{\epsilon_n}{4}\right\}\right| < \delta\right\} \in F(I).$$

Since $A \cap B \in F(I)$ and $\emptyset \notin F(I)$, we can choose $m \in A \cap B$. Then

$$\frac{1}{\lambda_m}\left|\left\{k \in I_m : \|x_k^n - L_n\| \geq \frac{\epsilon_n}{4} \vee \|x_k^{n+1} - L_{n+1}\| \geq \frac{\epsilon_n}{4}\right\}\right| \leq 2\delta < 1.$$

Since $\lambda_m \to \infty$ and $A \cap B \in F(I)$ is infinite, we can actually choose the above m so that $\lambda_m > 5$ (say). Hence there must exist a $k \in I_m$ for which we have simultaneously, $\|x_k^n - L_n\| < \frac{\epsilon_n}{4}$ and $\|x_k^{n+1} - L_{n+1}\| < \frac{\epsilon_n}{4}$.

Then it follows that

$$\begin{aligned} \|L_n - L_{n+1}\| &\leq \|L_n - x_k^n\| + \|x_k^n - x_k^{n+1}\| + \|x_k^{n+1} - L_{n+1}\| \\ &\leq \|x_k^n - L_n\| + \|x_k^{n+1} - L_{n+1}\| + \|x - x^n\|_\infty + \|x - x^{n+1}\|_\infty \\ &< \frac{\epsilon_n}{4} + \frac{\epsilon_n}{4} + \frac{\epsilon_n}{4} + \frac{\epsilon_n}{4} = \epsilon_n. \end{aligned}$$

This implies that $\{L_n\}_{n\in N}$ is a Cauchy sequence in X and let $L_n \to L \in X$ as $n \to \infty$. We shall prove that $x \to L\ (S_\lambda(I))$. Choose $\varepsilon > 0$ and choose $n \in N$ such that $\epsilon_n < \frac{\epsilon}{4}$, $\|x - x^n\|_\infty < \frac{\epsilon}{4}$, $\|L_n - L\| < \frac{\epsilon}{4}$. Now since

$$\begin{aligned}\frac{1}{\lambda_\gamma}|\{k \in I_\gamma : \|x_k - L\| \geq \epsilon\}| &\leq \frac{1}{\lambda_\gamma}|\{k \in I_\gamma : \|x_k - x_k^n\| + \|x_k^n - L_n\| + \|L_n - L\| \geq \epsilon\}| \\ &\leq \frac{1}{\lambda_\gamma}\left|\left\{k \in I_\gamma : \|x_k^n - L_n\| \geq \frac{\epsilon}{2}\right\}\right|,\end{aligned}$$

it follows that

$$\left\{\gamma \in N : \frac{1}{\lambda_\gamma}|\{k \in I_\gamma : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \subset \left\{\gamma \in N : \frac{1}{\lambda_\gamma}\left|\left\{k \in I_\gamma : \|x_k^n - L\| \geq \frac{\epsilon}{2}\right\}\right| \geq \delta\right\}$$

for any given $\delta > 0$. This shows that $x \to L(S_\lambda(I))$ and this completes the proof of the theorem. □

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